Form A Solutions

Clover Math

April 2024

1. Three in five clovers will be flowers, and there are $\frac{35}{5} = 7$ groups of five clovers in the field. Therefore, the number of flowers in the field is $3 \cdot 7 = \boxed{21}$.

2. If you add the two orders, you get $1 + 2 = 3$ sandwiches and $2 + 1 = 3$ drinks. Therefore, we can just add the given costs to find that of three sandwiches and three drinks, which gives us $8.50 + 11.50 = \boxed{20}$.

3. We test out possible values of n that work for each of the two conditions and then pick the first one that works.

   For $n + 5$ to be divisible by 7, n has to be numbers such as 2, 9, 16, 23, 30, etc.

   For $n + 7$ to be divisible by 5, n has to be numbers such as 3, 8, 13, 18, 23, etc.

   It seems like $\boxed{23}$ is the smallest number shared by the two sequences, so it is our answer.

4. It would be a bad idea to multiply out all of those exponents. Instead, note that

   $$2^{2024} - 2^{2020} = 2^4(2^{2020} - 2^{2016}).$$

   The $2^{2020} - 2^{2016}$ term gets divided out, so we are left with $2^4 = \boxed{16}$.

5. Let's express Norbit's age as N, his brother's age as B, and his dad's age as D. Right now, we have

   $$N = B + 5 = \frac{D}{3}.$$

   Seven years ago, Norbit's age was $N - 7$ and his brother's age was $B - 7$, and the given information tells us that

   $$N - 7 = 2(B - 7) \implies B + 5 - 7 = 2B - 14 \implies B = 12.$$

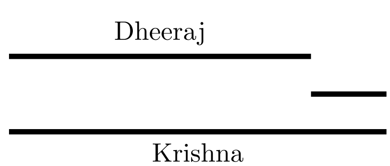
   Then, $D = 3(B + 5) = 3(12 + 5) = 3(17) = \boxed{51}$ years old.

6. Note the symmetry that results because the line passes through the center; you should find that the two pieces are congruent! The difference in areas is $\boxed{0}$.

7. Since $165 = 3 \cdot 5 \cdot 11$, the integers $3, 5$, and 11 must be our side lengths to avoid any length being equal to 1. The surface area is therefore

   $$2 \cdot (3 \cdot 5) + 2 \cdot (3 \cdot 11) + 2 \cdot (5 \cdot 11) = \boxed{206}.$$

8. When Krishna and Dheeraj meet, their combined distances traveled will be twice the pool length, or 50 yards. To travel this combined distance, they are traveling at a combined speed of $2 + 3 = 5$ yards



   per second. Thus, they will take $\frac{50}{5} = \boxed{10}$ seconds to meet.

9. For any positive integer n, we can expand $n \cdot n!$ as

$$n \cdot n! = (n+1)! - n!.$$

Applying this to each term in the sequence, we obtain the telescoping sum

$$7! - 6! + 6! - 5! + 5! - 4! + 4! - 3! + 3! - 2! + 2! - 1! = 7! - 1! = \boxed{5039}.$$

10. Recall that a number is divisible by three as long as the sum of its digits are divisible by 3. Since $1+2+2+2+5 = 12$ is divisible by three, all permutations of 12225 will be divisible by three. Therefore, as long as the number is divisible by 4, it will be divisible by 12. The last two digits can be 12 or 52, with 3 ways to order the remaining three digits. The answer is $2 \cdot 3 = \boxed{6}$.

11. Notice that the triangle with the dotted lines is $\frac{1}{4}$ the area of the entire triangle. (To see this, the dotted lines have half the length of AB and AC because the midline separates two congruent triangles). By Heron's formula, the area of the triangle is $\sqrt{21 \cdot 6 \cdot 7 \cdot 8} = 84$, and so the area of the new figure is $84 - \frac{1}{4}84 = \boxed{63}$.

12. For each prime number, we look for the highest power of the prime among the integers from 1 to 10. For 2, that's $8 = 2^3$. For 3, that's $9 = 3^2$. For 5 and 7, the highest exponent is 1. Thus, the answer is $3 + 2 + 1 + 1 = \boxed{7}$.

13. Firstly, $n > 6$ since otherwise 16_n would be undefined. Then, its guessing and checking until you find a working n. $n = 7$ does not work since $21_7 = 15$, which is not prime. Trying $n = 11$, we see that $21_{11} = 23$, which is prime, and $16_{11} = 17$, which is also prime. Seems like $n = \boxed{11}$ is our answer.

14. We can build a range of duck numbers from the given information. The minimum number of ducks is 55, which would occur if Jet used 11 5-duck nets in the second case, and the maximum number of ducks is 60, which would occur if Jet used 10 6-duck nets in the third case. If we switched one of the 6-duck nets in the third case with a 3-duck net, we would have 57 ducks; switch another, however, and we would have 54, which is outside of our range. Therefore, we have $\boxed{57}$ ducks.

15. First, note that

$$\sqrt{4^{4^4}} = ((2^2)^{4^4})^{1/2} = 2^{\frac{1}{2} \cdot 2 \cdot 4^4} = 2^{4^4} = 2^{2^8}.$$

This is a bit clearer since everything is now in powers of two. Solving for x gives

$$8 = 2^3 = 2^x \implies x = \boxed{3}$$

16. We first count the light grey unit squares to get the area they make up, which is 20.

    There are eight black quarter-circles with radius 2, which adds up to an area of $8 \cdot 1/4 \cdot 4\pi = 8\pi$.

    There are eight more dark grey quarter-circles with radius 1, which adds up to an area of $8 \cdot 1/4 \cdot \pi = 2\pi$.

    Adding up all these results, we get a total area of $\boxed{20 + 10\pi}$.

17. The total area of the target is that of the outermost circle, or 49π.

    The area of the 7-point region is π, giving a $\frac{1}{49}$ probability of getting 7 points.

    The area of the 5-point region is $9\pi - \pi = 8\pi$, giving an $\frac{8}{49}$ probability of getting 5 points

    The area of the 3-point region is $25\pi - 9\pi = 16\pi$, giving a $\frac{16}{49}$ probability of getting 3 points.

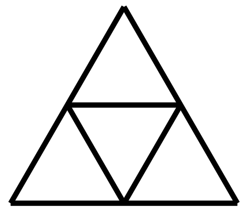
    The area of the 1-point region is $49\pi - 25\pi = 24\pi$, giving a $\frac{24}{49}$ probability of getting 1 point.

    Then, the expected score from the shot is

$$7 \cdot \frac{1}{49} + 5 \cdot \frac{8}{49} + 3 \cdot \frac{16}{49} + 1 \cdot \frac{24}{49} = \frac{119}{49} = \frac{17}{7}.$$

This gives us an answer of $17 + 7 = \boxed{24}$.

18. We use these 6 points to form a large equilateral triangle made from 4 smaller equilateral triangles, as depicted in the diagram. This makes $\boxed{5}$ total equilateral triangles.

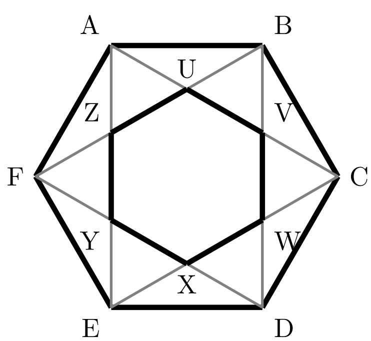


19. We expect there to be a pattern, so we list out some of the first turns and their corresponding candy counts.

Person's Turn	**Red Candies**	**Blue Candies**
Initial Counts	50	50
Bobby	49	50
Joe	49	47
Bobby	48	47
Joe	45	47
Bobby	44	47
Joe	44	44
...	...	...

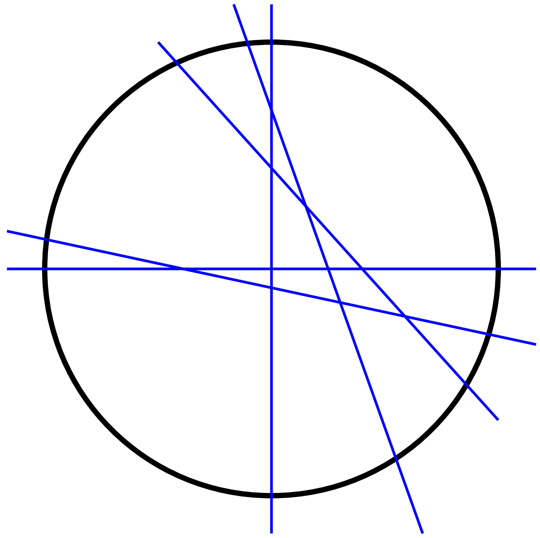
So, it seems like we get back to equal numbers of candy after having reduced both counts by 6. During this process, there are 3 instances where there are more blue than red candies. We do this the 8 times it takes to get to 2 red and 2 blue. Then, Bobby eats once so that there are 2 blue and 1 red and Joe, who is unable to continue eating, stops the two from continuing. This gives us $3 * 8 + 1 = \boxed{25}$ moments where there are more blue than red candies.

20. Let's label the inner hexagon $UVWXYZ$. Notice that triangles BUV, CVW, DWX, EXY, FYZ, and AZU are all equilateral; this tells us that $UV = \frac{1}{3}AC$, etc. Since $AC = \sqrt{3}AB$, the inner hexagon's side lengths are $\frac{1}{\sqrt{3}}$ those of the outer hexagon's side length; the inner hexagon has a third of the area of the outer hexagon. The outer hexagon has an area of $54\sqrt{3}$, so the inner hexagon has an area of $\boxed{18\sqrt{3}}$.



21. We make the cuts so that no three of them intersect at the same point but that all of the cuts intersect with each other inside the pizza. You could draw this and simply count up the number of regions. One such cut arrangement is shown:

Alternatively, notice that drawing the $n+1$th line cuts through $n+1$ of the regions formed by the first n lines; basically, this is the sequence we use to form triangular numbers. 5 cuts means the 5th triangular number, which is 15, plus 1 for there being one piece (the whole pizza) originally. Either way you do this problem, the answer is $\boxed{16}$.



22. Working backward is simpler than working forward. The last number is 1, so the 2nd to last is 2; since the 3rd to last number would be one if it was odd, it instead has to be 4. We then alternate between odd and even for the rest of the sequence, so when we list out the numbers from last to first, we get

$$1, 2, 4, 3, 6, 5, 10, 9, 18, 17, \ldots$$

Notice that the 2nd number is one more than 2^0, the 4th number is one more than 2^1, and so on until the 20th number, which is one more than 2^9. There are 20 numbers, which means $\boxed{19}$ seconds pass before the number 1 is written.

23. We might have noticed that we can use Simon's Favorite Factoring Trick to get

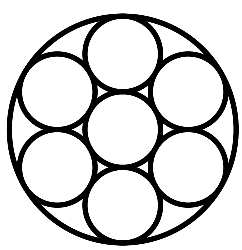
$$a \triangle b = ab - a - b = (a-1)(b-1) - 1.$$

Now notice that

$$(170-1)(26-1) - 1 = 169 \cdot 25 - 1 = (65-1)(65+1) = 64 \cdot 66$$

Thus, $27 * (170 \triangle 26) = 28 \cdot 64 \cdot 66 = 2^9 \cdot 3 \cdot 7 \cdot 11$, which has $10 \cdot 2 \cdot 2 \cdot 2 = \boxed{80}$ factors.

24. We place a basketball in the center and have a hexagonal arrangement of basketballs around it, as depicted in the diagram below. This gives us $\boxed{7}$ balls.



25. We anticipate that there will be repetition, so we list out the first few numbers in the sequence.

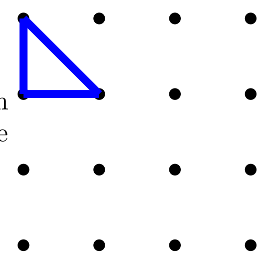
$$a_1 = 2024 \to a_2 = \frac{a_1}{a_1 - 1} = \frac{2024}{2023} \to a_3 = \frac{a_2}{a_2 - 1} = 2024$$

Indeed, the repetition is there. For odd n, $a_n = 2024$; for even n, $a_n = \frac{2024}{2023}$. Since 2024 is even, $a_{2024} = \boxed{\frac{2024}{2023}}$.

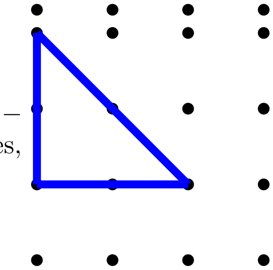
26. For the left hand-side to be divisible by 3, both x and y must be multiples of 3 (can you see why?). Then, let $x = 3a$ and $y = 3b$. Plugging these in and simplifying, we obtain $3a^2 + 3b^2 = z^2$, implying that z is a multiple of 3. Then let $z = 3c$, and plugging it in we get $a^2 + b^2 = 3c^2$, the same equation! If we repeat the process, we will get an infinite number of these equations, none of them yielding any solutions. The only solution that works then is $(0, 0, 0)$, so there is $\boxed{1}$ solution. As a side note, this type of technique is called infinite descent.

27. We will use the Principle of Inclusion-Exclusion. There are $\frac{10!}{2!2!}$ total permutations. Now we count the number of permutations that, when reversed, l is in the same spot. There are 5 pairs of spots for the l's to be, and then $\frac{8!}{2!}$ ways to order the remaining letters. Repeating the process for e, we get $2 \cdot 5 \cdot \frac{8!}{2!} = 5 \cdot 8!$ total ways. However, we overcount when both l and e are in the same position when reversed, to which there are $5 \cdot 4 = 20$ ways to order the l's and e's, times $6!$ ways to order the remaining 6 letters. Thus, our answer is $\frac{\frac{10!}{2!2!} - 5 \cdot 8! + 20 \cdot 6!}{6!} = 1260 - 280 + 20 = \boxed{1000}$.

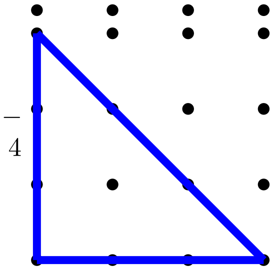
28. This problem is all about casework: we want to find the different possible sizes of isosceles right triangles that exist and how many of each size can be created.



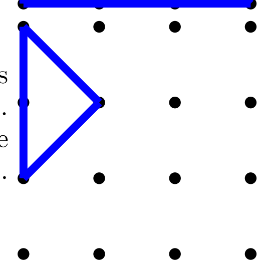
Let's start off simple, with a $1 - 1 - \sqrt{2}$ side length triangle. In every unit square, we can make 4 of these triangles, and there are 9 unit squares in the grid. This gives us 36 cases.



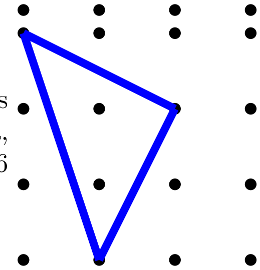
We can do something similar with triangles of side lengths $2 - 2 - 2\sqrt{2}$. In every 2x2 square, we can make 4 of these triangles, and there are 4 such 2x2 squares. This gives us 16 cases



Now for the largest triangles, which have side lengths of $3 - 3 - 3\sqrt{2}$. These require the full grid size to form, so there are only 4 cases.



We are not done yet. Note that the leg lengths of the isoceles right triangles don't necessarily need to be integer side lengths. We can do, for example, $\sqrt{2} - \sqrt{2} - 2$ side length triangles. We can fit 2 of these in every 2x1 or 1x2 area, of which there are 12. This gives us 24 cases.



Finally, the trickiest case to deal with. We can make triangles with side lengths of $\sqrt{5} - \sqrt{5} - \sqrt{10}$. In every 2x3 or 3x2 area, we can fit 4 of these, and there 4 such regions. This gives us 16 cases.

Summing all of these up, we get $\boxed{96}$ possible isosceles right triangles.

29. This question is solved mainly through intuition and experimentation. First, observe that k has no prime factors that are not factors of n. Obviously, if n has one prime factor, k's options are very limited. So we try when n has two factors first. Trying $n = 12$, we find that $k = 1, 2, 3, 4, 6, 8, 12$. Trying $n = 18$, we find that k can be $1, 2, 3, 4, 6, 8, 9, 18$. Trying $n = 24$, we strike jackpot because k can be $1, 2, 3, 4, 6, 8, 9, 12, 16, 24$. Now we consider the case when n has 3 prime factors. The smallest n can be is 30, which is already too large. Our answer is $\boxed{24}$.

30. Let S be the sum we are seeking. Then

$$\begin{aligned}
S &= \sum_{n=1}^{\infty} \frac{F_n}{2^n} \\
&= \frac{0}{2} + \frac{1}{4} + \sum_{n=3}^{\infty} \frac{F_n}{2^n} \\
&= \frac{1}{4} + \sum_{n=1}^{\infty} \frac{F_n + F_{n+1}}{2^{n+2}}. \\
&= \frac{1}{4} + \frac{1}{4}\sum_{n=1}^{\infty} \frac{F_n}{2^n} + \frac{1}{2}\sum_{n=1}^{\infty} \frac{F_{n+1}}{2^{n+1}} \\
&= \frac{1}{4} + \frac{1}{4}S + \frac{1}{2}S \\
&\implies S = \frac{1}{4} + \frac{3}{4}S \\
&\implies S = \boxed{1}.
\end{aligned}$$